Fractional, maximal and singular operators in variable exponent Lorentz spaces

by

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Dedicated to Professor Anatoly Kilbas on the occasion of his 60th birthday

Abstract

We introduce the Lorentz space $\mathcal{L}^{p(\cdot),q(\cdot)}$ with variable exponents $p(t), q(t)$ and prove the boundedness of singular integral and fractional type operators, and corresponding ergodic operators in these spaces. The main goal of the paper is to show that the boundedness of these operators in the spaces $\mathcal{L}^{p(\cdot),q(\cdot)}$ is possible without the local log-condition on the exponents, typical for the variable exponent Lebesgue spaces; instead the exponents $p(s)$ and $q(s)$ should only satisfy decay conditions of log-type as $s \to 0$ and $s \to \infty$. To prove this, we base ourselves on the recent progress in the problem of the validity of Hardy inequalities in variable exponent Lebesgue spaces.

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1 Introduction

Nowadays the so called variable exponent analysis is a popular topic which continues to attract many researchers, both in view of possible applications and also because of difficulties in investigation and existing challenging problems. This topic is mainly focused on the Lebesgue and Sobolev spaces with variable order of integrability and operator theory in these spaces. In particular, various results on non-weighted and weighted boundedness in Lebesgue spaces with variable exponents $p(x)$ have been proved for maximal, singular and fractional type operators, we refer to surveying papers [4], [11], [16]. As is well known, these boundedness results in the case of a bounded open set in $\mathbb{R}^n$ hold under the assumption that the exponent satisfies everywhere the local log-condition

$$|p(x) - p(y)| \leq \frac{A}{\ln \frac{1}{|x-y|}}, \tag{1.1}$$

for all $x, y \in \Omega$ with $|x-y| \leq \frac{1}{2}$. In the case of unbounded sets in $\mathbb{R}^n$, it is also supposed that there exists the limit $p(\infty) = \lim\limits_{\Omega \ni x \to \infty} p(x)$ and the decay condition of log-type

$$|p(x) - p(\infty)| \leq \frac{C}{\ln(e + |x|)} \tag{1.2}$$

is satisfied. Conditions (1.1)-(1.2) are known to be necessary, in terms of continuity moduli, for the boundedness of the maximal operator in the spaces $L^{p(\cdot)}(\Omega)$ with variable exponent $p(x)$, see [3], [15]. Since the known means to study singular and fractional operators in variable exponent spaces are somehow related to the maximal operator, assumptions (1.1)-(1.2) are always inherited, when one deals with those operators.

The goal of this note is to show that in the case of the Lorentz spaces $\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, when $p(t), q(t)$ are functions of $t \in \mathbb{R}^1_+$, the local log-condition (1.1) is no more needed for the boundedness of the maximal operator in $\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, we may use only decay conditions at two points, at $t = 0$ and $t = \infty$:

$$|p(t) - p(0)| \leq \frac{C}{\ln |t|} \quad \text{for} \quad |t| \leq \frac{1}{2}, \quad \text{and} \quad |p(t) - p(\infty)| \leq \frac{C}{\ln(e + |t|)}. \tag{1.3}$$

We base ourselves on a recent result [5] on the validity of the one-dimensional Hardy inequalities under assumptions of type (1.3).

The spaces $\mathcal{L}^{p(\cdot)}(\Omega) = \mathcal{L}^{p(\cdot),p(\cdot)}(\Omega)$ have already been introduced, see [12], where the boundedness of singular and fractional type operators was obtained under the assumption that the local log-condition (1.1) holds. Making use of the progress for the Hardy inequalities in [5], we now are able to avoid that condition and admit Lorentz spaces $\mathcal{L}^{p(\cdot),q(\cdot)}(\Omega)$.

2 Definitions

2.1 On variable exponent Lebesgue spaces and Hardy operators

Let Ω be an open set in $\mathbb{R}^n$ and μ a Borel measure on Ω. Let $p(x)$ be a μ-measurable function on Ω such that $1 \leq p_- := \operatorname{ess\,inf} p(x) \leq p_+ := \operatorname{ess\,sup} p(x) < \infty$. By $L^{p(\cdot)}(\Omega)$ we denote the space of measurable functions $f(x)$ on Ω such that

$$\mathfrak{I}_p(f) = \int_\Omega |f(x)|^{p(x)} d\mu(x) < \infty.$$

This is a Banach function space with respect to the norm

$$\|f\|_{L^{p(\cdot)}} = \inf \left\{ \lambda > 0 : \mathfrak{I}_p\left(\frac{f}{\lambda}\right) \leq 1 \right\}$$

(see e.g. [6]). We refer to [2] for definition and fundamental properties of Banach function spaces.

We denote $\frac{1}{p'(x)} = 1 - \frac{1}{p(x)}$.

In the one-dimensional case $n = 1$ we deal with the interval $[0, \ell]$, $0 < \ell \leq \infty$ and the standard Lebesgue measure. Let

$$p_- = \inf_{t \in [0,\ell]} p(t), \qquad p_+ = \sup_{t \in [0,\ell]} p(t).$$

We will use the notation

$$\mathcal{P}_a = \{p : a < p_- \leq p_+ < \infty\}, \quad a \in \mathbb{R}^1 \tag{2.1}$$

and will be interested in the special cases of the classes $\mathcal{P}_a$ with $a = 0$ or $a = 1$.

Definition 2.1. By $\mathbb{P}([0, \ell])$ we denote the class of functions $p \in L^\infty([0, \ell])$ such that there exist the limits

$$p(0) = \lim_{t \to 0} p(t) \quad \text{and} \quad p(\infty) = \lim_{t \to \infty} p(t),$$

and conditions (1.3) are satisfied, the conditions at infinity being only needed in the case $\ell = \infty$. We also denote

$$\mathbb{P}_a([0, \ell]) = \mathbb{P}([0, \ell]) \cap \mathcal{P}_a([0, \ell]).$$

We recall that for $p \in \mathcal{P}_1([0, \ell])$ the Hölder inequality

$$\left| \int_0^\ell u(t)v(t)dt \right| \leq k\|u\|_{L^{p(\cdot)}} \|v\|_{L^{p'(\cdot)}} \tag{2.2}$$

holds with $k = \frac{1}{p_-} + \frac{1}{p'_-}$.

3

In [5] the following statement was proved.

Theorem 2.2. *Let $p \in \mathbb{P}_1([0, \ell])$ and $\alpha, \beta, \nu \in \mathbb{P}([0, \ell])$ and*

$$0 \leq \nu(0) < \frac{1}{p(0)} \quad and \quad 0 \leq \nu(\infty) < \frac{1}{p(\infty)}. \tag{2.3}$$

Let also $q(x)$ be any function in $\mathbb{P}_1([0, \ell])$ such that

$$\frac{1}{q(0)} = \frac{1}{p(0)} - \nu(0) \quad and \quad \frac{1}{q(\infty)} = \frac{1}{p(\infty)} - \nu(\infty). \tag{2.4}$$

Then the Hardy-type inequalities

$$\left\| t^{\alpha(t)+\nu(t)-1} \int_0^t \frac{f(s)\,ds}{s^{\alpha(s)}} \right\|_{L^{q(\cdot)}([0,\ell])} \leq C \left\| f \right\|_{L^{p(\cdot)}([0,\ell])} \tag{2.5}$$

$$\left\| t^{\beta(t)+\nu(t)} \int_t^\ell \frac{f(s)\,ds}{s^{\beta(s)+1}} \right\|_{L^{q(\cdot)}([0,\ell])} \leq C \left\| f \right\|_{L^{p(\cdot)}([0,\ell])}, \tag{2.6}$$

are valid, if and only if

$$\alpha(0) < \frac{1}{p'(0)}, \quad \alpha(\infty) < \frac{1}{p'(\infty)} \tag{2.7}$$

and

$$\beta(0) > -\frac{1}{p(0)}, \quad \beta(\infty) > -\frac{1}{p(\infty)}, \tag{2.8}$$

respectively (conditions at the point ∞ in (2.3)-(2.4) and (2.7)-(2.8) being only required in the case $\ell = \infty$)

2.2 Variable exponent Lorentz spaces

In the sequel we denote $\ell = \mu\Omega$ for brevity. On the base of the Lebesgue $L^{p(\cdot)}([0, \ell])$ we introduce now some new Banach function spaces, variable exponent Lorentz spaces. By

$$f^*(t) = \sup\{s \geq 0 : \mu(\{x \in \Omega : |f(x)| > s\}) > t\}$$

we denote the non-increasing rearrangement of a function f. Obviously $f^*(t) \equiv 0$ for $t > \ell$ in case $\ell < \infty$.

Definition 2.3. Let $p, q \in \mathcal{P}_0([0, \ell])$. By $\mathcal{L}^{p(\cdot), q(\cdot)}(\Omega)$ we denote the space of functions f on Ω such that $t^{\frac{1}{p(t)} - \frac{1}{q(t)}} f^*(t) \in L^{q(\cdot)}([0, \ell])$, i.e.

$$\mathfrak{I}_{p,q}(f) := \int_0^\ell t^{\frac{q(t)}{p(t)}-1} |f^*(t)|^{q(t)} \, dt < \infty, \tag{2.9}$$

4

and we use the notation

$$\|f\|_{\mathcal{L}^{p,q}(\Omega)} = \inf\left\{\lambda > 0 : \mathfrak{I}_{p,q}\left(\frac{f}{\lambda}\right) \le 1\right\} = \left\|t^{\frac{1}{p(t)}-\frac{1}{q(t)}}f^*(t)\right\|_{L^{q(\cdot)}([0,\ell])}. \qquad (2.10)$$

It is easy to see that in the case $p \in \mathbb{P}_0([0,\ell]), q \in \mathbb{P}_1([0,\ell])$, condition (2.9) is equivalent to the condition

$$\int_0^1 t^{\frac{q(0)}{p(0)}-1}|f^*(t)|^{q(t)}\,dt + \int_1^\infty t^{\frac{q(\infty)}{p(\infty)}-1}|f^*(t)|^{q(t)}\,dt < \infty, \qquad (2.11)$$

the latter being written for the case $\ell = \infty$. In the case $\ell < \infty$, only the term $\int_0^\ell t^{\frac{q(0)}{p(0)}-1}|f^*(t)|^{q(t)}\,dt$ should be considered.

Let

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds, \qquad f^*(t) \le f^{**}(t).$$

We can introduce the norm

$$\|f\|^1_{\mathcal{L}^{p,q}(\Omega)} = \left\|t^{\frac{1}{p(t)}-\frac{1}{q(t)}}f^{**}(t)\right\|_{L^{q(\cdot)}([0,\ell])}, \qquad (2.12)$$

so that

$$\|f\|_{\mathcal{L}^{p,q}(\Omega)} \le \|f\|^1_{\mathcal{L}^{p,q}(\Omega)}.$$

The equivalence of (2.10) and (2.12) is characterized in the following theorem.

Theorem 2.4. *Let $p \in \mathbb{P}_0([0,\ell]), q \in \mathbb{P}_1([0,\ell])$. Then the inequality $\|f\|^1_{\mathcal{L}^{p,q}(\Omega)} \le C\|f\|_{\mathcal{L}^{p,q}(\Omega)}$ with a constant $C > 0$ not depending on f, holds if and only if $p(0) > 1$ and, in case the $\ell = |\Omega| = \infty$, also $p(\infty) > 1$.*

Proof. Indeed, the inequality $\|f\|^1_{\mathcal{L}^{p,q}(\Omega)} \le C\|f\|_{\mathcal{L}^{p,q}(\Omega)}$ is nothing else but the boundedness in $L^{q(\cdot)}([0,\ell])$ of the Hardy operator

$$t^{\frac{1}{p(t)}-\frac{1}{q(t)}-1}\int_0^t \frac{f(s)ds}{s^{\frac{1}{p(s)}-\frac{1}{q(s)}}}.$$

By Theorem 2.2, this boundedness is valid if and only if the values of $\frac{1}{p(t)} - \frac{1}{q(t)}$ at the points $t = 0$ and $t = \infty$ are less than those of $\frac{1}{q'(t)}$ at these points, respectively. This gives conditions $p(0) > 1, p(\infty) > 1$. $\qquad \square$

Note that in all the statements in the sequel, all the conditions imposed on $p(t), q(t)$ at the point $t = \infty$ should be omitted in the case where $|\Omega| < \infty$.

In accordance with Theorem 2.4, in the sequel we consider the space $\mathcal{L}^{p(\cdot),q(\cdot)}(\Omega)$ under the following assumptions on $p(\cdot)$ and $q(\cdot)$:

$$p, q \in \mathbb{P}_1([0,\ell]) \quad \text{and} \quad p(0) > 1, \ p(\infty) > 1. \qquad (2.13)$$

2.3 Basic properties of the spaces $\mathcal{L}^{p,q}(\Omega)$

We refer to [2] for the notion of Banach function space (BFS) and rearrangement invariant norms, but recall the following basic definition, where $M(\Omega, \mu)$ denotes the set of all μ-measurable functions on Ω.

Definition 2.5. A normed linear space $X = (X(\Omega, \mu), \| \ \|_X)$ is called a Banach function space, if the following conditions are satisfied:
i) the norm $\|f\|_X$ is defined for all $f \in M(\Omega, \mu)$;
ii) $\|f\|_X = 0$ if and only if $f(x) = 0$ μ-a.e. on Ω;
iii) $\|f\|_X = \||f|\|_X$ for all $f \in X$;
iv) for every $Q \subset \Omega$ with $\mu Q < \infty$ we have $\|\chi_Q\|_X < \infty$;
v) if $f_n \in M(\Omega, \mu)$, $n = 1, 2, \ldots$ and $f_n \nearrow f$ μ-a.e. on Ω, then $\|f_n\|_X \nearrow \|f\|_X$;
vi) if $f, g \in M(\Omega, \mu)$ and $0 \le f(x) \le g(x)$ μ-a.e. on Ω, then $\|f\|_X \le \|g\|_X$;
vii) given $Q \subset \Omega$ with $\mu Q < \infty$, there exists a constant c_Q such that for all $f \in X$, $\int_Q |f(x)| d\mu \le c_Q \|f\|_X$.

In particular, the following statement is known ([1], p.61).

Proposition 2.6. *Let (X, μ) be an arbitrary totally σ-finite measure space and $\lambda(g)$ a rearrangement-invariant norm over $(\mathbb{R}^1, m)$. Then the functionional $\rho(f)$ defined on functions f in (X, μ) by $\rho(f) = \lambda(f^*)$ is a rearrangement-invariant norm on (X, μ).*

Lemma 2.7. *Let $p, q \in \mathcal{P}_1(\Omega)$. Then the dual space $\left(\mathcal{L}^{p(\cdot),q(\cdot)}(\Omega)\right)^*$ is $\mathcal{L}^{p'(\cdot),q'(\cdot)}(\Omega)$.*

Theorem 2.8. *Under conditions (2.13), the space $\mathcal{L}^{p,q}(\Omega)$ is a Banach function space.*

Proof. To state that both $\|f\|_{\mathcal{L}^{p,q}(\Omega)}$ and $\|f\|^1_{\mathcal{L}^{p,q}(\Omega)}$ are norms, it suffices to refer to Proposition 2.6. (The triangle inequality for the norm $\|f\|^1_{\mathcal{L}^{p,q}(\Omega)}$ follows from the inequality $(f+g)^{**}(t) \le f^{**}(t) + g^{**}(t)$, see e.g. [10], Section 2, or [1], p. 54). The other requirements to the definition of BFS easily follow from properties of non-increasing rearrangements f^* and properties of the spaces $L^{p(\cdot)}$. For example, iv) is valid since for $0 \le f_n \nearrow f$ we have $f_n^* \nearrow f^*$ (see e.g. [18], Lemma 3.5, Chapter 5). Then

$$\|f_n\|_{\mathcal{L}^{p,q}(\Omega)} = \left\| t^{\frac{1}{p(t)} - \frac{1}{q(t)}} f_n^* \right\|_{L^{q(\cdot)}([0,\ell])} \nearrow \|f\|_{\mathcal{L}^{p,q}(\Omega)}$$

by the property of the space $L^{q(\cdot)}$. To check vii), we make use of the Hölder inequality (2.2) for $L^{q(\cdot)}$ with $u(t) = t^{\frac{1}{q(t)} - \frac{1}{p(t)}}$ and $v(t) = t^{\frac{1}{p(t)} - \frac{1}{q(t)}} f^*(t)$ and get

$$\int_Q |f(x)| dx = \int_0^{\mu Q} f^*(t) dt \le \|u\|_{L^{q'(\cdot)}([0,\ell])} \|f\|_{L^{p(\cdot),q(\cdot)}(\Omega)} \le c_Q \|f\|_{L^{p(\cdot),q(\cdot)}(\Omega)}$$

with $c_Q = \|u\|_{L^{q'(\cdot)}([0,\ell])} < \infty$ because $\|u\|_{L^{q'(\cdot)}([0,\ell])} < \infty \iff \mathfrak{I}_q(u) < \infty$, the latter being valid under the condition $p(0) > 1$, which was assumed. $\qquad \square$

Let $w(t)$ be a nonnegative weight function defined on $[0, \ell]$.

Definition 2.9. We define the weighted Lorentz space $\mathcal{L}_w^{p(\cdot),q(\cdot)}(\Omega)$ with the weight w defined on $[0, \ell]$, as the subset of functions in $M(\Omega, \mu)$ such that

$$\|f\|_{\mathcal{L}_w^{p(\cdot),q(\cdot)}(\Omega)} = \left\| w(t) t^{\frac{1}{p(t)} - \frac{1}{q(t)}} f^*(t) \right\|_{L^{q(\cdot)}(\Omega)} < \infty. \tag{2.14}$$

Let also

$$\|f\|^1_{\mathcal{L}_w^{p(\cdot),q(\cdot)}(\Omega)} = \left\| w(t) t^{\frac{1}{p(t)} - \frac{1}{q(t)}} f^{**}(t) \right\|_{L^{q(\cdot)}(\Omega)}. \tag{2.15}$$

In the next lemma we suppose that $\gamma(t)$ is a measurable bounded function on $[0, \ell]$ having the limit $\gamma(0) = \lim\limits_{t \to 0+} \gamma(t)$, and, in the case $\ell = \infty$, also having the limit $\gamma(\infty) = \lim\limits_{t \to +\infty} \gamma(t)$ and satisfying the conditions

$$|\gamma(t) - \gamma(0)| \leq \frac{C}{\ln \frac{1}{t}}, \quad 0 < t < \frac{1}{2} \quad \text{and} \quad |\gamma(t) - \gamma(\infty)| \leq \frac{C}{\ln(e + t)}. \tag{2.16}$$

Lemma 2.10. *Let the conditions in* (2.13) *be satisfied and let* $w(t) = t^{\gamma(t)}$, *where* $\gamma(t)$ *satisfies conditions* (2.16) *and*

$$\gamma(0) < \frac{1}{p'(0)} \quad \text{and} \quad \gamma(\infty) < \frac{1}{p'(\infty)}.$$

Then $\|f\|_{\mathcal{L}_w^{p(\cdot),q(\cdot)}(\Omega)} \leq \|f\|^1_{\mathcal{L}_w^{p(\cdot),q(\cdot)}(\Omega)} \leq C\|f\|_{\mathcal{L}_w^{p(\cdot),q(\cdot)}(\Omega)}$, *where* $C > 0$ *does not depend on* f.

Proof. The left hand side inequality is trivial, the right-hand side one follows from Theorem 2.2.

$\square$

In the next theorem we use the notation

$$\mathcal{L}_{loc}^{p(\cdot)}([0, \ell]) = \left\{ f : f \in \mathcal{L}^{p(\cdot)}([0, \ell_1]) \text{ for all } \ell_1 < \ell \right\}.$$

Theorem 2.11. *Under the condition*

$$\frac{t^{\frac{1}{q(0)} - \frac{1}{p(0)}}}{w(t)} \in \mathcal{L}_{loc}^{q'(\cdot)}([0, \ell]), \tag{2.17}$$

the space $\mathcal{L}_w^{p(\cdot),q(\cdot)}(\Omega)$ *is a Banach function space with respect to the norm* $\|f\|^1_{\mathcal{L}_w^{p(\cdot),q(\cdot)}(\Omega)}$.

The proof is similar to that of Theorem 2.8.

3   On classical operators in the space $\mathcal{L}_w^{p(\cdot),q(\cdot)}(\Omega)$

Let

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{\mu B(x, r)} \int\limits_{\Omega \cap B(x,r)} |f(y)| d\mu(y), \quad x \in \Omega, \tag{3.18}$$

be the Hardy-Littlewood maximal function.

Theorem 3.12. *Let p and q satisfy assumptions (2.13). Then the maximal operator is bounded in the space $\mathcal{L}_w^{p(\cdot),q(\cdot)}(\Omega)$ with the weight*

$$w(t) = t^{\gamma(t)}, \quad \gamma \in \mathbb{P}([0,\ell]), \tag{3.19}$$

if

$$\gamma(0) < \frac{1}{p'(0)} \quad and \quad \gamma(\infty) < \frac{1}{p'(\infty)} \quad (\text{the latter in the case} \quad \mu(\Omega) = \infty). \tag{3.20}$$

Proof. As is known,

$$(\mathcal{M}f)^*(t) \le Cf^{**}(t), \tag{3.21}$$

see for instance [2], p.122. Therefore,

$$\|\mathcal{M}f\|_{\mathcal{L}_w^{p(\cdot),q(\cdot)}(\Omega)} = \left\| t^{\gamma(t)+\frac{1}{p(t)}-\frac{1}{q(t)}}(\mathcal{M}f)^* \right\|_{L^{q(\cdot)}([0,\ell])} \le C \left\| t^{\gamma(t)+\frac{1}{p(t)}-\frac{1}{q(t)}} f^{**} \right\|_{L^{q(\cdot)}([0,\ell])} \tag{3.22}$$

and then the result follows by Theorem 2.2. $\square$

As is known, the identity approximations

$$A_\varepsilon f(x) = \frac{1}{\varepsilon^n} \int_{\mathbb{R}^n} a\left(\frac{x-y}{\varepsilon}\right) f(y),$$

where $\int_{\mathbb{R}^n} a(y)\, dy = 1$ and $a(x)$ has a radial decreasing integrable majorant, are dominated by the maximal operator:

$$|A_\varepsilon f(x)| \le C\mathcal{M}f(x), \quad f \in L^p(\mathbb{R}^n),\ 1 \le p \le \infty, \tag{3.23}$$

with an absolute constant $C > 0$ not depending on x and ε, see [17]. In particular, the Poisson integral

$$\mathbf{P}_y f(x) = \int_{\mathbb{R}^n} P\left(x-\xi, y\right) f(\xi)d\xi, \quad P(x,y) = \frac{c_n y}{(|x|^2 + y^2)^{\frac{n+1}{2}}}, \quad y > 0$$

is uniformly in y dominated by the maximal function. Under assumptions of Theorem 3.12 we have

$$\mathcal{L}_w^{p(\cdot),q(\cdot)}(\Omega) \subset L^1(\Omega) + L^\infty(\Omega). \tag{3.24}$$

So we make use of (3.23) and arrive at the following corollary.

Corollary 3.13. *Under the assumptions of Theorem 3.12, the sublinear operator*

$$\sup_{\varepsilon > 0} |A_\varepsilon f(x)|,$$

where $A_\varepsilon f$ is an identity approximation with kernel admitting radial decreasing integrable majorant, is bounded in the space $\mathcal{L}_w^{p(\cdot),q(\cdot)}(\Omega)$; in particular the operator $\sup_{y>0} |\mathbf{P}_y f(x)|$ is bounded in this space.

Next we consider in $\mathcal{L}_w^{p(\cdot),q(\cdot)}(\Omega)$ convolution operators

$$k * f(x) = \int_{\mathbb{R}^n} k(x-y)f(y)d\mu(y).$$

We will also treat their particular cases, the Riesz potential operator and Calderon-Zygmund singular operators, which for generality we will consider over an open set $\Omega \subseteq \mathbb{R}^n$:

$$I^\alpha f(x) = \int_\Omega \frac{f(y)}{|x-y|^{n-\alpha}} \, d\mu(y), \quad x \in \Omega, \quad 0 < \alpha < n.$$

and

$$Kf(x) = \int_\Omega \frac{A(x-y)}{|x-y|^n} f(y) \, d\mu(y), \quad x \in \Omega,$$

where A is an odd function on $\mathbb{R}^n$, homogeneous of degree 0 and satisfying the Dini condition on the unit sphere $\mathbb{S}^{n-1}$:

$$\int_0^2 \frac{\omega(A,\delta)}{\delta} \, d\delta < \infty, \quad \text{where} \quad \omega(k,\delta) = \sup_{x,y \in S^{n-1}, |x-y| \leq \delta} |A(x) - A(y)|.$$

The operators K include as particular cases, the Hilbert transform $(n = 1, \ k(x) = \frac{x}{|x|})$ and the Riesz transforms $(n \geq 2, \ k(x) = \frac{x_j}{|x|}, \ j = 1, \ldots, n)$.

There are known the following pointwise estimates of those classical operators via decreasing rearrangements:

$$(k * f)^*(t) \leq k^{**}(t) \int_0^t f(s)ds + \int_t^\infty k^*(s)f^*(s)ds, \tag{3.25}$$

see [13], and its particular case

$$(I^\alpha f)^*(t) \leq c\left(t^{-1+\alpha/n} \int_0^t f^*(s)ds + \int_t^\ell f^*(s)s^{-1+\alpha/n}ds\right), \quad \ell = \mu(\Omega). \tag{3.26}$$

A similar estimate holds for the singular operator K

$$(Kf)^*(t) \leq c\left(\frac{1}{t} \int_0^t f^*(s)ds + \int_t^\ell \frac{f^*(s)}{s} \, ds\right), \quad \ell = \mu\Omega, \tag{3.27}$$

see [1].

Theorem 3.14. *Let p and q satisfy assumptions (2.13). Then the operator K is bounded in the space $\mathcal{L}_w^{p(\cdot),q(\cdot)}(\Omega)$ with the weight (3.19) under conditions (3.20).*

9

Proof. The proof is obtained similarly to (3.22) from the pointwise estimate (3.27) and Theorem 2.2. □

Theorem 3.15. *Let $0 < \alpha < n$, p and q satisfy assumptions (2.13) and $p_+ < \frac{n}{\alpha}$. Then the operator I^α is bounded from the space $\mathcal{L}_w^{p(\cdot),q(\cdot)}(\Omega)$ with the weight (3.19) into the space $\mathcal{L}_w^{p_\alpha(\cdot),q(\cdot)}(\Omega)$ where $\frac{1}{p_\alpha(t)} = \frac{1}{p(t)} - \frac{\alpha}{n}$, if*

$$\frac{\alpha}{n} - \frac{1}{p(0)} < \gamma(0) < \frac{1}{p'(0)} \quad and \quad \frac{\alpha}{n} - \frac{1}{p(\infty)} < \gamma(\infty) < \frac{1}{p'(\infty)}, \qquad (3.28)$$

the condition at infinity being needed in the case $\mu(\Omega) = \infty$.

Proof. We have

$$\|I^\alpha f\|_{\mathcal{L}_w^{p_\alpha(\cdot),q(\cdot)}(\Omega)} = \left\| t^{\gamma(t)+\frac{1}{p_\alpha(t)}-\frac{1}{q(t)}} \left(I^\alpha f \right)^* (t) \right\|_{L^{q(\cdot)}([0,\ell])}.$$

Then by (3.26)

$$\|I^\alpha f\|_{\mathcal{L}_w^{p_\alpha(\cdot),q(\cdot)}(\Omega)} \le c(A + B),$$

where

$$A = \left\| t^{\lambda(t)-1} \int_0^t \frac{\varphi(s)ds}{s^{\lambda(s)}} \right\|_{L^{q(\cdot)}([0,\ell])}, \quad B = \left\| t^{\lambda(t)-\frac{\alpha}{n}} \int_0^t \frac{\varphi(s)ds}{s^{\lambda(s)-\frac{\alpha}{n}+1}} \right\|_{L^{q(\cdot)}([0,\ell])}$$

and $\lambda(t) = \gamma(t) + \frac{1}{p(t)} - \frac{1}{q(t)}$ and $\varphi(t) = t^{\lambda(t)} f^*(t) \in L^{q(\cdot)}([0,\ell])$. It remains to make use of Theorem 2.2. □

Since the fractional maximal function

$$\mathcal{M}^\alpha f(x) = \sup_{r>0} \frac{1}{|B(x,r)|^{1-\frac{\alpha}{n}}} \int_{B(x,r)\cap\Omega} |f(y)|dy, \quad 0 < \alpha < n,$$

is dominated by fractional integral: $\mathcal{M}^\alpha f(x) \le c\, I^\alpha(|f|)(x)$, from Theorem 3.15 we get the following corollary,

Corollary 3.16. *Under the assumptions of Theorem 3.15, the operator $\mathcal{M}^\alpha$ is bounded from the space $\mathcal{L}_w^{p(\cdot),q(\cdot)}(\Omega)$ into the space $\mathcal{L}_w^{p_\alpha(\cdot),q(\cdot)}(\Omega)$.*

4 On the ergodic maximal function and the ergodic Hilbert transform in variable exponent Lorentz spaces

Let $(T_\tau)_{\tau\in\mathbb{R}}$ be an ergodic flow of measure-preserving transformations on a σ-finite measure space (X,μ), and let $\mathbf{M}f$ and $\mathbb{H}f$, $f \in L(X)$, be the ergodic maximal function and the ergodic Hilbert transform, respectively, (see [14])

$$\mathbf{M}f(x) = \sup_{a>0} \frac{1}{a} \int_0^a |f(T_\tau x)|\, d\tau \quad and \quad \mathbb{H}f(x) = \lim_{\delta\to 0+} \int_{\{\delta\le|t|\le 1/\delta\}} \frac{f(T_\tau x)}{\tau}\, d\tau.$$

10

The estimations (3.21) and (3.27) hold for operators $\mathbf{M}$ and $\mathbb{H}$, respectively, as well. Namely,

$$(\mathbf{M}f)^*(t) \le f^{**}(t) \tag{4.29}$$

can be obtained as in the discrete case (see [7]; Ineq. (2)) since only the weak $(1,1)$ type inequality, $\mu\{(\mathbf{M}f)^* > \lambda\} \le \frac{1}{\lambda} \int_{\{(\mathbf{M}f)^* > \lambda\}} f \, d\mu$, is used to prove (4.29) in the discrete case which holds for the continuous case too with equation sign (see [14], p. 76), and the inequality

$$(\mathbb{H}f)^*(t) \le c\left(\frac{1}{t} \int_0^t f^*(s)ds + \int_t^\ell \frac{f^*(s)}{s} \, ds\right), \quad \ell = \mu(X), \tag{4.30}$$

can be proved using the generalization of the Stein-Weiss theorem for the ergodic Hilbert transform (see [8], [9]):

$$\mu\{|\mathbb{H}(\mathbf{1}_E)| > \lambda = \begin{cases} \Psi_{\mu(E)}(\lambda) & \text{when } \mu(X) = \infty \\ \Phi_{\mu(E)}(\lambda) & \text{when } \mu(X) < \infty, \end{cases} \tag{4.31}$$

where $E \subset X$ is any measurable subset, and

$$\Psi_\xi(\lambda) = \frac{2\xi}{\sinh\lambda} \quad \text{and} \quad \Phi_\xi(\lambda) = \frac{2\mu(X)}{\pi} \arctan\left(\frac{\sin(\pi\xi/\mu(X))}{\sinh\lambda}\right).$$

Indeed, if h is a measurable function with strictly decreasing continuous distribution function $\mathcal{D}_h$, then $h^*(t) = \mathcal{D}_h^{-1}(t)$. Hence it follows from (4.31) that

$$\left(\mathbb{H}(\mathbf{1}_E)\right)^*(t) = \begin{cases} \Psi_{\mu(E)}^{-1}(t) & \text{when } \mu(X) = \infty \text{ and } 0 < t < \infty \\ \Phi_{\mu(E)}^{-1}(t) & \text{when } \mu(X) < \infty \text{ and } 0 < t < \mu(X) \\ 0 & \text{when } \mu(X) < \infty \text{ and } t \ge \mu(X) \end{cases}$$

Observe that

$$\Psi_{\mu(E)}^{-1}(t) = \sinh^{-1}\left(\frac{\xi}{t}\right) \quad \text{and} \quad \Phi_{\mu(E)}^{-1}(t) = \sinh^{-1}\left(\frac{\sin(\pi\xi/\mu(X))}{\tan(\pi t/2\mu(X))}\right).$$

The function $\sinh^{-1}$ is increasing, and if we use simple relations between the trigonometric functions $\sin x < x$, $0 < x < \pi$ and $\tan t > t$, $0 < t < \frac{\pi}{2}$, then we get for each $\mu(E) < \mu(X)$ and $t > 0$,

$$(\mathbb{H}(\mathbf{1}_E))^*(t) \le \frac{1}{\pi} \sinh^{-1}\left(\frac{2\mu(E)}{t}\right) \tag{4.32}$$

The rest of the proof of (4.30) is the same as for the usual Hilbert transform case (see [14], pp.134-137).

As in previous sections, depending on estimations (4.29) and (4.30), one can prove the following

Theorem 4.17. *Let p and q satisfy assumptions (2.13). Then the ergodic maximal operator and the ergodic Hilbert transform are bounded in the space $\mathcal{L}_w^{p(\cdot),q(\cdot)}(\Omega)$ with the weight (3.19) under conditions (3.20).*

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